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FACTORING WEAKLY COMPACT OPERATORS

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The aim of this note is to discuss the structure of weakly compact operators. We extend results well known for Banach spaces to operators acting on Banach spaces.

Let E and F be two Banach spaces and let $T \in L(E, F)$. Consider the following two conditions on T:

(wP) T is weakly precompact, i.e. T maps bounded sequences into sequences with weak Cauchy subsequences;

(D) T maps weak Cauchy sequences into weakly convergent sequences.

Each weakly compact operator verifies (wP) and (D) and the product of an operator having property (D) with an weakly precompact operator is weakly compact. Moreover, the main result in [1] asserts that indeed each weakly compact operator can be obtained in such a way, so the interest in studying the two above classes (actually ideals in the sense of Pietsch) of operators.

H. P. Rosenthal [11] has obtained a nice characterization of weakly precompact operators in terms of basic sequences:

1. Theorem. *An operator $T \in L(E, F)$ is weakly precompact if and only if T does not fix a copy of l_1.*

T is said to fix a copy of the Banach space X provided that T is an isomorphism when restricted to some subspace of E, isomorphic to X.

Other characterizations can be obtained by easy modifications of some results due to Pelczynski [8] who considered only the case when T is the identity of a separable Banach space.

2. Theorem. *Let E be a separable Banach space, F a Banach space and $T \in L(E, F)$. Then the following assertions are equivalent:*

i) *T fixes a copy of l_1;*

ii) *T' fixes a copy of $C[0, 1]'$;*

iii) *T' fixes a copy of $l_1(\Gamma)$ for some uncountable set Γ.*

The proof of i) $\Rightarrow$ ii) follows from [8], while iii) $\Rightarrow$ i) can be adapted from [13].

H. P. Rosenthal [10] has obtained an interesting dichotomy for subspaces A of a space $L_1(\mu)$, related to condition iii) in Theorem 2 above: either there exists an $f \in L_1(\mu)$ such that $A \subset L_1(\lambda)$, where $d\lambda = f d\mu$, or A contains a subspace complemented in $L_1(\mu)$ and isomorphic to $l_1(\Gamma)$ for some uncountable set Γ. This result has also an operatorial companion.

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3. Theorem. *Let E be a Banach lattice with a weak order unit $u > 0$, F a Banach space and $T \in L(E, F)$. Then*

i) *either T' fix a copy of $l_1(\Gamma)$ for some uncountable set Γ; or*

ii) *the image of T' is contained in the band $[u']$ generated by a suitable $u' \in E'$.*

The two possibilities listed above are disjoint if E' is in addition weakly sequentially complete.

See also [6] for a more precise result.

Proof. Suppose that ii) fails. Then there are a uncountable family of pairwise disjoint elements $u'_\gamma \in E'$ and a family $y'_\gamma \in F'$, $\gamma \in \Gamma$, such that

$$P_\gamma(T'y'_\gamma) \neq 0.$$

Here P_γ denotes the canonical projection of E' onto $[u'_\gamma]$. Clearly, we may assume that $\| y'_\gamma \| = \| u'_\gamma \| = 1$ and (by passing to a uncountable subset of Γ if necessary) that

$$| P_\gamma(T'y'_\gamma)u | > \delta$$

for each γ. Consider the operator $S : E' \to l_1(\Gamma)$ given by

$$Sx' = \{P_\gamma(x')u\}_\gamma, \; x' \in E'.$$

Then by Lemma 1.1 in [10], $S \cdot T'$ is an isomorphism when restricted to some subspace of F' isomorphic to $l_1(\Gamma)$.

If E is a Banach lattice with E' weakly sequentially complete then the spaces $[u']$ are weakly compactly generated, while $l_1(\Gamma)$ is not if Γ is uncountable. ■

If E is a separable Banach lattice with E' weakly sequentially complete then the subspaces of E' contained in subspaces $[u']$ are precisely the separable ones. This fact combined with the main result of [5] and Theorems 2 and 3 above provides new characterizations of weakly precompact operators:

4. Theorem. *Let E be a separable Banach lattice with E' weakly sequentially complete, F a Banach space and $T \in L(E, F)$. Then the following assertions are equivalent*

i) *T is weakly precompact;*

ii) *T does not fix a copy of* C[0, 1];

iii) Im *T' is separable.*

As a corollary we reobtain Lotz's characterization of dual Banach lattices having the Radon-Nikodym property:

5. Corollary. *The following assertions are equivalent for E a separable Banach lattice:*

i) *E does not contain a copy of l_1;*

ii) *E does not contain a complemented copy of* C[0, 1];

iii) *E does not contain a copy of* C[0, 1];

iv) *E does not contain a copy of* $L_1[0, 1]$;

v) *E' is separable;*

vi) *E' is weakly compactly generated.*

The restrictions on E in Theorem 4 above cannot be dropped without additional hypotheses on T. Here are two counterexamples.

The first one concerns the separability of E. Consider for T the identity of $c(\Gamma)$ where Γ is a uncountable set. Then E is nonseparable (but has a strong order unit), E' is weakly sequentially complete, T is weakly precompact and Im T' is nonseparable.

The second counterexample concerns the restrictions on E'. R. C. James has constructed in [3] an example of a separable Banach space JT with nonseparable dual and such that each infinite dimensional subspace of JT contains a copy of l_2. Let $\pi : l_1 \to$ JT an onto mapping. Then π is weakly precompact, defined on a separable Banach lattice and Im π' is nonseparable.

There is yet another characterization of weakly precompact operators due to Odell, important in producing compact operators. An operator from a Banach space to another is called a *Dunford-Pettis* (D. P.) *operator* provided it maps weak Cauchy sequences into norm convergent sequences. Odell's characterization asserts that an operator $T \in L(E, F)$ is weakly precompact iff for each D.P. operator $S \in L(F, G)$, the composition SoT is compact. See [12] for details.

The condition (D) was first considered by Grothendieck [2] who obtained conditions under which an operator verifying (D) is weakly compact.

If an operator T has property (D) then T does not fix a copy of c_0 and verifies also the *Pelczynski's property* (u), i.e. for each weak Cauchy sequence $(x_n)_n \subset E$ there is a weakly summable sequence $(y_n)_n \subset \overline{\text{Im } T}$ such that

$$Tx_n - \sum_{k=1}^{n} y_k \xrightarrow{w} 0.$$

The later two conditions are independent. Consider for example the identity of c_0 and the identity of the James'space J (which fails (u)). However, under additional hypotheses the fact that T does not fix a copy of c_0 implies that T has property (D).

6. Theorem. *Let $T \in L(E, F)$ an operator which does not fix a copy of c_0. Then T has the property* (D) *in each of the following cases:*

i) *E is isomorphic to an AM-space;*

ii) *F does not contain a copy of c_0 and T has property* (u);

iii) *E is isomorphic to a Banach lattice with an order continuous norm.*

The following question is open: Let E be a Banach lattice and $T \in L(E, E)$ an operator wich does not fix a copy of c_0 or l_1. Is T^2 weakly compact?

Here is an example which shows that T need not be weakly compact. Let $\sigma : l_1 \to c$ given by $\sigma((\alpha_n)_n) = \left(\sum_{k=1}^{n} \alpha_k\right)_n$. Then $T = \sigma \oplus 0 : l_1 \oplus c \to c \oplus l_1$ is not weakly compact and $T^2 = 0$.

The property (D) was studied in [7] in connection with the following two classes of operators T defined on Banach lattices E and taking values in arbitrary Banach spaces F:

T is said to be of *type A* provided that T is order σ-continuous, i.e.

$0 \leqslant x_n \downarrow$ in E implies $(Tx_n)_n$ is norm convergent in F.

T is said to be of *type B* provided that

$0 \leqslant x_n \uparrow$, $\| x_n \| \leqslant K$ in E implies $(Tx_n)_n$ is norm convergent in F.

Property (D) implies type B which in turn implies type A. It was noted that property (D) (called there strong type B) is equivalent to the fact that T'' maps the band B (generated by E in E'') into F.

In the sequel we shall study the duality between (wP) and type A.

7. Theorem. *Let E be a σ-complete Banach lattice with a weak order unit $u > 0$, F a Banach space and $T \in L(E, F)$.*

Then T is of type A iff T can be factored through a weakly compactly generated Banach space.

Proof. Suppose that T is of type A. Then, as noted in [7], T maps each order interval into a relatively weakly compact subset of F a and thus $X = \overline{\text{Span}\, T[-u, u]}$ is a weakly compactly generated subspace of F. On the other hand $x = \sup(x \wedge nu)$ and $T(x \wedge nu) \to Tx$ for each $x \in E^+$. Consequently $X \supset T(E)$.

Conversely, if T can be factored through a weakly compactly generated space then T does not fix a copy of l_∞. Indeed, l_∞ is not weakly compactly generated and any complemented subspace of a weakly compactly generated Banach space so is weakly compactly generated. Consequently (see [7], Lemma 3.1) T is of type A. ■

The result of Theorem 6 fails if we drop the assumption on the existence of a weak order unit. See the case when $T = 1_{l_1(\Gamma)}$ for Γ a uncountable set.

Amir and Lindenstrauss have proved (see [1] for details) that the unit ball of the dual of a weakly compactly generated Banach space is w'-sequentially compact. Consequently :

8. Corollary. *Let E be a σ-complete Banach lattice with a weak order unit $u > 0$, F a Banach space and $T \in L(E, F)$.*

Then T is of type A iff T' maps bounded sequences into sequences with w'-convergent subsequences.

The next result extends a well known fact due to Pelczynski.

9. Proposition. *Let E and F be two Banach spaces and $T \in L(E, F)$. Then the following assertions are equivalent*

i) *T' fixes a copy of c_0;*

ii) *T' fixes a copy of l_∞;*

iii) *There is a complemented subspace X of E, isomorphic to l_1, such that $T \mid X$ is an isomorphism and $T(X)$ is complemented in F.*

Proof. Clearly, we have only to show that i) ⇒ iii). For, let $(y'_n)_n$ be a basic sequence in F' which is equivalent to the natural basis of c_0 and let i be the canonical inclusion of $Y = \overline{\text{Span}(y'_n)_n}$ into F'. If $T' \circ i$ is an isomorphism then $i' \circ T'' \mid E$ verifies the assumptions of Lemma 1.1 in [10], which yields the required X. ■

We comes now to the duality between (wP) and type A.

10. Theorem. *Let E and F be two Banach lattices and $T \in L(E, F)$.*

i) *If F' has a weak order unit then T is weakly precompact iff T' is of type A.*

ii) *If E is σ-complete and F' has a weak order unit then T is weakly precompact iff T and T' are of type A.*

Proof. i) If T is weakly precompact then by Proposition 8 above T' fix no copy of l_∞, which implies (see Lemma 3.1 in [7]) that T' is of type A.

If T' is of type A then by Corollary 7 above $T'' \mid E$ maps bounded sequences into sequences with w'-convergent subsequences in F'', i.e. into weak Cauchy sequences in F.

ii) If E is σ-complete and T is weakly precompact then T fix no copy of l_∞ (in fact, $l_\infty \supset l_1$) which implies (see Lemma 3.1 in [7]) that T is of type A. ■

REFERENCES

[1] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.

[2] A. Grothendieck, *Sur les applications lineaires faiblement compactes d'espaces du type C(K)*, Canad. Journal Math. **5** (1951), 129–173.

[3] R. C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. A.M.S. **80** (1974), 738–743.

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes in Math. n° 338, Springer-Verlag, 1973.

[5] H. P. Lotz and H. P. Rosenthal, *Embeddings of $C(\Delta)$ and $L^1[0, 1]$ in Banach lattices*, Israel J. Math. **31** (1978), 169–179.

[6] C. Niculescu, *Banach lattices with a weak order unit*, Bull. A.M.S. **82** (1976), 748–750.

[7] C. Niculescu, *Order σ-continuous operators*, to appear.

[8] A. Pelczynski, *On Banach spaces containing L_1 [0, 1]*, Studia Math. **30** (1968), 231–246

[9] A. Pelczynski, *On C(S) subspaces of separable Banach spaces*, Studia Math. **31** (1968) 513–522.

[10] H. P. Rosenthal, *On injective Banach spaces and the spaces L^∞ (μ) for finite measures* Acta Math. **124** (1970), 205–248.

[11] H. P. Rosenthal, *A characterization of Banach spaces containing l_1*, Proc. Nat. Acad. Sc USA **71** (1974), 2411–2413

[12] H. P. Rosenthal, *Point-wise Compact Subsets of the first Baire Class*, Amer. Journal Math. **99**, 2 (1977), 362–378.

[13] C. Stegall, *Banach spaces whose duals contain $l_1(\Gamma)$*, Trans. Amer. Math. Soc. **176** (1976 463–477.

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